Method of complex paths and general covariance of Hawking radiation

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We apply the technique of complex paths to obtain Hawking radiation in different coordinate representations of the Schwarzschild space-time. The coordinate representations we consider do not possess a singularity at the horizon unlike the standard Schwarzschild coordinate. However, the event horizon manifests itself as a singularity in the expression for the semi-classical action. This singularity is regularized by using the method of complex paths and we find that Hawking radiation is recovered in these coordinates indicating the covariance of Hawking radiation. This also shows that there is no correspondence between the particles detected by the model detector and the particle spectrum obtained by the quantum field theoretic analysis – a result known in other contexts as well.

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Quantum field theory(QFT) requires the notion of a time-like killing vector field to define particles. When curvilinear coordinate transformations are allowed in flat space time, there exists other possible killing vector fields which are time-like in part of the manifold. Particle states – in particular vacuum state – can be defined with respect to these killing vector fields. In general, these definitions will not be equivalent and the Minkowski vacuum will appear to be a many particle states according to the new definition. Thus, the presence of, say, Rindler quanta in the Minkowski vacuum arises because there is more than one way of defining positive modes in a given space-time, even though the space-time itself is static. On the other hand, particles are created in a time dependent metric because the natural definition of positive frequency modes are different at two different times. This is interpreted as the production of particles corresponding to quantum field by the changing geometry of spacetime.

The most spectacular prediction of quantum field theory in curved space time is undoubtedly the Hawking radiation from a black hole. Hawking [1] showed that QFT in the background of a body collapsing to a black hole predicts a radiation of particles at late times with characteristic thermal spectrum at temperature equal to $1/8\pi M$. But, as mentioned in the previous paragraph, the concept of a particle in quantum field theory is not generally covariant and depends on the coordinates chosen to describe the particular space-time. It is therefore of interest to study the Hawking effect in other coordinate representations. The method used by Hawking in his analysis requires the knowledge of the wave modes of the quantum field in the standard Schwarzschild coordinate(SSC). In attempting to study Hawking effect in other coordinate systems, one runs into problem of identifying the wave modes and calculating Bogoluibov coefficients to identify the spectrum of the radiation which is intractable. Hence, it is necessary to have a method which does not use wave modes to calculate the emission spectrum.

Hartle and Hawking [2] obtained particle production in the standard black hole space-times using semiclassical analysis which does not require wave modes. In this method, the semiclassical propagator for a scalar field propagating in the maximally extended Kruskal manifold is analytically continued in the time variable t to complex values. This analytic continuation gives the result that the probability of emission of particles from the past horizon is not the same as the probability of absorption into the future horizon. The ratio between these probabilities is of the form

$$P[\text{emission}] = P[\text{absorption}]e^{-\beta E}, \tag{1}$$

where E is the energy of the particles and $\beta^{-1} = 1/8\pi M$ is the standard Hawking temperature. The above relation is interpreted to be equivalent to a thermal distribution of particles in analogy with that observed in any system interacting with black body radiation. In this analysis, the probability amplitude for the emission/absorption is calculated by identifying a particular path for both these processes in the fully extended Kruskal manifold.

Unfortunately the Kruskal extension is of vital importance in obtaining the thermal spectrum in this analysis, and hence, it cannot be adapted for other coordinate systems. In Ref. [3], the authors have obtained Hawking radiation *without using the Kruskal extension*. They have shown that the coordinate singularity present at the horizon, in the SSC, manifests itself as a singularity in the expression for the semi-classical propagator $K(r_2, t_2; r_1, t_1)$, which is given by

$$K(r_2, t_2; r_1, t_1) = N \exp\left[iS_0(r_2, t_2; r_1, t_1)/\hbar\right]. \tag{2}$$

S_0 is the action functional satisfying the classical Hamilton-Jacobi equation for a massless particle to propagate from (t_1, r_1) to (t_2, r_2) and N is the suitable normalization constant. The authors used the method of complex paths (discussed in Ref. [4]) which was modified appropriately to produce a prescription that regularizes the singularity in the action functional and Hawking radiation was recovered as a consequence. The semi-classical propagator obtained in this fashion is an exact propagator of the quantum field.

In this letter, we apply the method of complex paths to two non-static coordinate representations of the Schwarzschild space-time and recover Hawking radiation. The two coordinates we consider are: Lemaitre coordinate(LC) which is a time dependent system and Painleve coordinate(PC) which is a stationary system. The line element corresponding to LC is,

$$ds^2 = d\tau^2 - \left[\frac{3}{4M}(R \pm \tau)\right]^{-2/3} dR^2 - \left[\sqrt{2M}\frac{3}{2}(R \pm \tau)\right]^{4/3} d\Omega^2, \quad (3)$$

where $c = G = 1$ and $d\Omega^2$ is the angular line element. The line elements can be modeled as that natural to a freely falling observer whose velocity at radial infinity is zero. The time coordinate τ measures the proper time of free falling observers; each observer moves along a line $R = constant$. The lower sign can be represented as particle trajectories moving inward to the singularity($r = 0$), while the upper sign represents that of moving outward from the singularity.

The line element corresponding to PC is,

$$ds^2 = (d\tau_P)^2 - \left[dr \pm \sqrt{\frac{2M}{r}} d\tau_P\right]^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (4)$$

The physical interpretation of these metrics is obtained by comparing them with LC. Noting that the constant time slice is a flat Euclidean space for the Painleve metric, we transform the coordinate variables of Lemaitre as

$$\tau_P = \tau, \quad r = (2M)^{1/3} \left[3(R - \tau)/2\right]^{2/3} \quad (5)$$

to go from (3) to (4). Under these transformations, the lower sign in line element (3) leads to (4) with upper sign; repeating the same steps starting from upper sign in (3) leads to the other line element.

Before we proceed with the analysis of quantum fields, it is necessary to understand certain important differences between the non static representations, LC and PC, with that of SSC. In both these representations, LC and PC, metric possesses no coordinate singularity at the horizon while SSC possesses a coordinate singularity at the horizon. The SSC is time reversal invariant ($t \to -t$), while the two coordinates, LC and PC, are not. In these two coordinates, the time reversal transformation on the metric corresponds to a different physical situation. However, the line elements corresponding to these different physical situations are related to the SSC by a coordinate transformation. The two line elements of Painleve metric can be obtained by the coordinate transformation

$$\tau_P = t \pm 2\sqrt{2Mr} \pm 2M \ln\left(\frac{\sqrt{r} - \sqrt{2M}}{\sqrt{r} + \sqrt{2M}}\right). \quad (6)$$

where, t is the time coordinate of SSC (the coordinate transformation between LC and SSC can be found in Ref. [5]).

In principle, calculating the probability amplitude for the emission/absorption process in these space-times requires identifying a particular path for both these processes, as done in Ref. [2], in the maximally extended Kruskal manifold which requires detailed knowledge of particle trajectories. It is, however, possible to analyze these two coordinate systems using the notion of R and T regions, introduced by Novikov [6], which provides an elegant method in understanding the global structure of the spacetime, thus making a detailed analysis of particle trajectories unnecessary. The general expression for the interval in a spherically symmetric field is,

$$ds^2 = e^{\nu(x^0,x^1)}(dx^0)^2 - e^{\lambda(x^0,x^1)}(dx^1)^2 - e^{\mu(x^0,x^1)}d\Omega^2. \quad (7)$$

If at the given event in the general coordinate system, the inequality,

$$e^{\nu-\lambda} > \left(\frac{\partial\mu}{\partial x^0} / \frac{\partial\mu}{\partial x^1}\right)^2 \quad (8)$$

is satisfied, then the event is defined as R-region. If the opposite inequality is satisfied, the event is in a T-region. The definitions of R and T regions can be shown to be coordinate invariant.

For the LC, we find that the region outside the Schwarzschild sphere [for the lower sign in line element (3)] the inequality $\frac{3}{2}(R - \tau) > 2M$ corresponds to R-region; while the region inside the Schwarzschild sphere, where the inequality is opposite to the above inequality, is a T-region. In the T-region there is an obvious asymmetry in the direction of time flow. For this line element, all motions in the T-region(T_-) are directed towards $r = 0$ where the curvature invariants are infinite. The upper sign in line element (3) will define second type of T-regions — an expanding region T_+. This has completely opposite properties and all bodies move away from the singularity to the R-regions. For a physically realisable event, the coexistence of T_+ and T_- regions is impossible. In other words, the T region of the standard Schwarzschild is a doubly mapped T region in Lemaitre.

In the case of PC, using the transformation relating the Lemaitre and Painleve in Eq. (5), we find that the inequality $r > 2M$ corresponds to R-region for both the line elements in (4). Hence, for both these line elements, R and T regions are the same — the whole of the spacetime is doubly mapped.

We will now proceed to study QFT in these coordinate systems. We are interested in the quantization of a massless scalar field Φ in these two backgrounds. The system we first consider is the lower sign in line element (4) of Painleve. Since all the relevant physics is contained in the (τ_P, r) plane, we set $\Phi = \Psi(\tau_P, r)Y_l^m(\theta, \phi)$ and concentrate on Ψ. The equation satisfied by Ψ is,

$$\frac{\partial^2\Psi}{\partial\tau_P^2} - \sqrt{\frac{8M}{r}}\frac{\partial^2\Psi}{\partial\tau_P\partial r} - \frac{3}{2r}\sqrt{\frac{2M}{r}}\frac{\partial\Psi}{\partial\tau_P} - \frac{2}{r}\left[1-\frac{M}{r}\right]\frac{\partial\Psi}{\partial r}$$
$$-\left(1-\frac{2M}{r}\right)\frac{\partial^2\Psi}{\partial r^2} - \frac{l(l+1)}{r^2}\Psi = 0. \qquad (9)$$

The semiclassical wave-functions satisfying the above equation are obtained by making the standard ansatz for Ψ *i.e.*,

$$\Psi(\tau_P, r) = \exp\left[iS(\tau_P, r)/\hbar\right] \qquad (10)$$

where S is a function which is expanded in powers of $\hbar$ of the form

$$S(r,\tau_P) = S_0(r,\tau_P) + \hbar\, S_1(r,\tau_P) + \hbar^2 S_2(r,\tau_P)\ldots \qquad (11)$$

Substituting into the wave equation (9) and neglecting terms of order $\hbar$ and greater, we find to the lowest order,

$$\left(1-\frac{2M}{r}\right)\left(\frac{\partial S_0}{\partial r}\right)^2 - \left(\frac{\partial S_0}{\partial\tau_P}\right)^2 + \sqrt{\frac{8M}{r}}\left(\frac{\partial S_0}{\partial\tau_P}\right)\left(\frac{\partial S_0}{\partial r}\right)$$
$$+ \frac{l(l+1)\hbar^2}{r^2} = 0 \qquad (12)$$

The above equation is just the Hamilton-Jacobi equation satisfied by a massless particle moving in the space time determined by the line element (4). The action functional(S_0) satisfying the classical Hamilton-Jacobi equation will immediately give us the semiclassical Kernel $K(r_2,t_2;r_1,t_1)$ for the particle to propagate from (t_1,r_1) to (t_2,r_2) in the saddle point approximation. Introducing a dimensionless variable $\rho = r/2M$ and considering the case $l=0$ for simplicity, the solution to the above equation is easily found to be,

$$S_0 = -E\tau_P + 2ME\int d\rho\frac{\sqrt{\rho}(1\pm\sqrt{\rho})}{\rho-1}. \qquad (13)$$

Choosing the positive sign in the above equation, we see that the solution is singular at $\rho = 1$ which corresponds to the horizon. The method of complex paths identifies appropriate complex contours which satisfies the semiclassical condition to be the one lying in the upper complex plane (For an extensive discussion on this, see [3]). The complex paths, we use in our analysis, takes into account of all possible paths satisfying the semi- classical ansatz - irrespective of the multiple mapping of the part/whole of the space-time. For the PC, as noted earlier, the whole of space-time is doubly mapped *w.r.t.* the Schwarzschild space-time implying that PC contains two distinct R and T regions. Hence, the semi-classical propagator will have contributions from complex paths from both of these regions which have no common point due to each path being in different R and T regions. Hence, the contribution to the amplitude of emission/absorption by these two paths will be mutually exclusive. Thus, regularizing the singularity by such contours and dividing the result by two to take care of the over counting of the paths, we obtain

$$S_0[\text{emission}] = \text{real part} + 2i\pi ME \qquad (14)$$

In this case, the action that has been calculated is interpreted to be that for emission by analogy with done in Ref. [3]. In order to obtain the action for absorption of particles, we have to consider the Hamilton-Jacobi equation for the upper sign in the line element (4) and repeating the above calculations and using the arguments above we obtain

$$S_0[\text{absorption}] = \text{real part} - 2i\pi ME \qquad (15)$$

where the minus sign arises from choosing the appropriate complex contour given by the semiclassical prescription. Taking the modulus square to obtain the probability, we get $P[\text{emission}] \propto \exp(-4\pi ME)$ and $P[\text{absorption}] \propto \exp(4\pi ME)$, thus

$$P[\text{emission}] = \exp(-8\pi ME)\, P[\text{absorption}]. \qquad (16)$$

The exponential dependence on the energy allows one to give a "thermal" interpretation to this result, which shows that the temperature of the emission spectrum is the standard Hawking temperature.

In Ref. [3], the authors have shown explicitly that very close to the horizon, the terms containing the mass and angular part does not contribute significantly. Similar results hold in our case and the above analysis is therefore applicable to both massless and massive scalar particles.

In our interpretation, we consider the amplitude for pair creation both inside and outside the horizon. The semiclassical treatment of Hawking radiation for the PC is obtained recently by Parikh and Wilczek [7]. The authors considered Hawking radiation as a pair creation outside the horizon, with the negative energy particle tunneling into the black hole. The tunneling of particles produced just inside the horizon also contributes to the Hawking radiation and is incorporated in our formalism.

We now consider the quantization of massless scalar field Φ in the LC. The system we first consider is the lower sign in line element (3). Separating the angular variables by setting $\Phi = \Psi(\tau,R)Y_l^m(\theta,\phi)$, making the standard semiclassical ansatz for Ψ and expanding S in powers of $\hbar$ one finds, to lowest order,

$$\left(1-U^{2/3}\right)\left[(\partial_U S_0)^2 + (\partial_V S_0)^2\right] - 2\left(1+U^{2/3}\right)(\partial_U S_0)$$
$$\times(\partial_V S_0) + \frac{16l(l+1)}{9U^{2/3}} = 0, \qquad (17)$$

where U and V are dimensionless parameters given by

$$U = \frac{3}{4M}(R-\tau), \qquad V = \frac{3}{4M}(R+\tau). \qquad (18)$$

The above equation is just the Hamilton-Jacobi equation satisfied by a massless particle moving in the space time determined by the line element (3). Specializing to the case $l = 0$ for simplicity, the solution to the above equation is easily found to be,

$$\frac{df}{dU} = E \frac{1 + U^{2/3} \pm 2U^{1/3}}{1 - U^{2/3}}, \tag{19}$$

where the $\pm$ signs arise from taking square roots. Notice that it is only for the positive sign that the denominator is singular at $U = 1$. This singular solution for f evidently corresponds to outgoing particles. Therefore choosing the positive sign and making the convenient change of variable $x^3 = U$ the solution to Eq. (17) is given by

$$S_0 = -\frac{4ME}{3} V + 4ME \int dx \frac{x^2(1+x)}{1-x}. \tag{20}$$

It is clear that the action function is again singular at the horizon $x = 1$. Using the method of complex paths, we regularize the singularity by a complex contour lying in the upper complex plane and obtain

$$S_0[\text{emission}] = \text{real part} + 8i\pi ME \tag{21}$$

To obtain the action for absorption of particles, we repeat the above calculation for the upper sign in line element (3). In this case, it is easy to see that the only singular solution corresponds to in-going particles and using the method of complex paths as in previous case, we obtain

$$S_0[\text{absorption}] = \text{real part} - 8i\pi ME. \tag{22}$$

As seen earlier, the complex paths takes into account of all possible paths satisfying the semi-classical ansatz - irrespective of the multiple mapping of the space-time. For the PC, the contribution to the amplitude for absorption/ emission by the complex paths are mutually exclusive as the paths are in two distinct R and T regions. In the case of LC, however, only part of the region(T) is doubly mapped, it is always possible to find one point that is common to the paths contributing to absorption/emission. Hence, these paths are not mutually exclusive in calculating amplitudes. These paths, on the other hand, will be mutually exclusive when one considers the probability amplitude. Hence, the probability amplitude for absorption/ emission obtained by taking the modulus square of the semi-classical wave function will have equal contributions from these two paths. Thus, the total contribution to the probability is from four similar paths. Taking this into account, by dividing S_0 by four, we obtain

$$P[\text{emission}] = \exp\left(-8\pi ME\right) P[\text{absorption}]. \tag{23}$$

The exponential dependence of the energy allows us to give the thermal interpretation to this result, which shows that the temperature of the emission spectrum is the standard Hawking temperature.

It is normally assumed that the evaporation process results from an instability of the vacuum in the presence of the background metric. The particles are produced at a constant rate suggesting that the Hawking radiation converts the mass of the black hole into energy, thereby decreasing the mass. The decrease in the black hole mass is a physical effect and should be independent of the coordinate transformations and hence *Hawking radiation should be covariant*. Here we have shown that this is indeed true by the method of complex paths.

The above results can be summarized as follows: The analysis of the space-time structure of the two coordinates, LC and PC, using R and T regions provides an elegant method in understanding the global structure of the space-time, thus making the detailed analysis of particle trajectories unnecessary. The complex paths takes into account all paths irrespective of the multiple mapping of the part/whole of the space- time. This is crucial to obtain the correct temperature associated with the Hawking radiation in these two coordinates. It has been shown by Davies [8] that a freely falling detector will see a particle spectrum different from the thermal spectrum. This shows that there is no correspondence between the particles detected by the model detector and the particle spectrum obtained by the field theoretic analysis – a result known in other contexts as well (see Ref. [9] and references therein).

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